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Irreducible representations of the extended Poincaré parasuperalgebra

J Niederle† and A G Nikitin‡

† Institute of Physics, Academy of Sciences of the Czech Republic, Na Slovance 2, Prague 8, Czech Republic

‡ Institute of Mathematics, Academy of Sciences of the Ukraine, 3 Tereshenkivs'ka Str., Kiev-4, Ukraine

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Abstract. We classify and explicitly construct all the irreducible hermitian representations of the extended Poincaré parasuperalgebra. These representations which include the representations of the usual extended Poincaré superalgebra as a particular case can serve as a group-theoretical foundation of parasupersymmetric quantum field theory, i.e. as a general viewpoint to reformulate the quantum field theory and quantum mechanics of identical particles on the general basis of paraquantization and supersymmetry.

1. Introduction

It is pretty well known that description of identical particles in classical and quantum physics is completely different. Whereas in classical physics the identical particles can be enumerated and individually tracked, in quantum physics, due to Heisenberg's uncertainty relations, the notion of particle trajectory loses sense and identical particles become completely indistinguishable. Due to this and superposition principle the functions describing states of more than one identical particle should be always either symmetric or antisymmetric with respect to exchange of identical particles. Moreover, the particles described by symmetric (antisymmetric) functions obey the Bose–Einstein (Fermi–Dirac) statistics. Due to the theorem about the connection between spin and statistics, particles obeying Fermi–Dirac statistics are associated with half-integer spins, i.e. they are fermions, and particles obeying Bose–Einstein statistics are associated with integer spins, i.e. they are bosons.

The description of a system of fermions or bosons in the so-called field quantization schemes (known as 'second quantization') is also quite different. The field operators for bosons (i.e. the corresponding creation and annihilation operators) satisfy commutation relations, whereas those for fermions satisfy anticommutation relations. Thus bosons and fermions obey different rules and at first sight appear to have completely different physical properties.

The question as to whether they exhaust all possibilities in nature or whether particles exist that satisfy other statistics and quantization rules has been raised many times, together with the question as to how bosons and fermions are connected and how they can be transformed into one another.

At present there are several approaches to clarifying the similarities between Fermi and Bose statistics. The first is associated with the so-called *parastatistics*. It was Gentile [1] in 1940 who first mentioned that there might be other statistics and quantization rules than just

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those found by Fermi and Dirac or Bose and Einstein. This was based on the fact that physical observables are bilinear forms of creation and annihilation operators but not these operators alone.

In 1950 Wigner [2] stressed another important fact, namely that the usual canonical commutation or anticommutation relations of field operators are only sufficient to derive the equation of motion of a given physical system but not necessary. The necessary conditions are expressed in terms of double commutation (anticommutation) relations among the field operators - relations which are much weaker than the usual canonical ones.

In 1953 Green [3] put Wigner's quantization and Gentile's statistics together and introduced the so-called parastatistics.

Let us recall that the creation and annihilation operators of para-fields satisfy the trilinear relations

$$\begin{aligned} &[[a_k^+, a_j]_\pm, a_m]_- = -2\delta_{km} a_l \\ &[[a_k, a_l]_\pm, a_m]_- = 0 \end{aligned} \qquad (1.1)$$

together with the condition for the existence of a unique no-particle state vacuum

$$\begin{aligned} &a_k \Phi_0 = 0 \qquad \text{for all } k \\ &a_k a_l^+ \Phi_0 = p\delta_{kl}\Phi_0 \qquad \text{for all } k, l. \end{aligned} \qquad (1.2)$$

Here a_k and a_l^+ respectively denote the annihilation and creation para-field operators, Φ_0 is the vacuum state, the upper (lower) sign corresponds to the para-Bose (para-Fermi) case and integer $p > 0$ is the order of a given parastatistics. It should be noted that relations $[[a_k^+, a_l^+]_\pm, a_m]$ can be derived by hermitian conjugation of $(1, 1)$ and by use of the Jacobi-like identities.

It is known [4] that the above parastatistics relations reduce for $p = 1$ to the usual relations for fermions and bosons, and that the limit $p \to \infty$ for para-Bose (para-Fermi) statistics yelds in some sense the Fermi (Bose) theories (for other properties of parastatistics, their generalizations, more recent developments, etc, see, e.g., [5]).

The second approach which clarifies the connections between fermions and bosons is associated with *supersymmetry*—a new kind of symmetry which transforms bosonic states into the fermionic ones and vice versa, realizes a factorization of the Dirac operator, and so on. It was first introduced 30 years ago in quantum field theory [6] (refer to [7] for reviews).

Supersymmetric quantum field theory was soon followed by supersymmetric quantum mechanics [8]. In 1988 this mechanics was generalized to a special *parasupersymmetric* one [9] which deals with bosons and para-fermions. Another approach to parasupersymmetric quantum mechanics, namely with positive-definite Hamiltonians was proposed in [10]. Since its inception parasupersymmetric quantum mechanics has been a topics of many papers (see, for example, [11] and references cited therein) then parasupersymmetric quantum field theory has appeared and begun to be discussed [12]. In contrast to supersymmetric quantum field theory, in which field operators satisfy the usual Bose–Einstein or Fermi–Dirac statistics, the field operators in parasupersymmetric quantum field theory satisfy parastatistics.

In [13, 14] the irreducible representations (IRs) of the simplest $N = 1$ (i.e. including only one parasupercharge) Poincaré parasuperalgebra were described.

In this paper (using the generalized Wigner method of induced representations) irreducible representations of the extended Poincaré parasuperalgebra $p(1, 3; N)$ (i.e. the Poincaré parasuperalgebra with an arbitrary number N of parasupercharges, which includes the internal symmetry algebra) are classified and explicitly constructed. These representations form a group-theoretical basis of parasupersymmetric quantum field theory with N parasupercharges, which is the general standpoint for reformulation of the quantum field theory and quantum mechanics of identical particles on the general basis of paraquantization and supersymmetry.

We shall see that some IRs of the extended Poincaré *parasuperalgebra* also appear simultaneously to be IRs of the extended Poincaré *superalgebra*, so that they bring a deeper insight into the usual supersymmetric quantum field theory.

2. The extended Poincaré parasuperalgebra

The extended Poincaré parasuperalgebra $p(1,3;N)$ includes ten generators $P_\nu,\ J_{\nu\sigma}$ of the Poincaré group satisfying the usual commutation relations

$$\begin{aligned}
&[P_\mu, P_\nu] = 0 \qquad [P_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}P_\sigma - g_{\mu\sigma}P_\nu)\\
&[J_{\mu\nu}, J_{\rho\sigma}] = i(g_{\mu\sigma}J_{\nu\rho} + g_{\nu\rho}J_{\mu\sigma} - g_{\mu\rho}J_{\nu\sigma} - g_{\nu\sigma}J_{\mu\rho})\\
&J_{\mu\nu} = -J_{\nu\mu} \qquad \mu,\nu = 0,1,2,3 \qquad g_{\nu\nu} = (1,-1,-1,-1)
\end{aligned} \tag{2.1}$$

and $4N$ parasupercharges $Q_A^j,\ \bar{Q}_A^j\ (A = 1,2,\ j = 1,2,\ldots,N)$ which satisfy the double commutation relations

$$\begin{aligned}
&[Q_A^i, [Q_B^j, Q_C^k]] = [\bar{Q}_A^i, [\bar{Q}_B^j, \bar{Q}_C^k]] = 0\\
&[Q_A^i, [\bar{Q}_B^j, Q_C^k]] = 4\delta_{ij} Q_C^k (\sigma_\mu)_{AB} P^\mu\\
&[\bar{Q}_A^i, [Q_B^j, \bar{Q}_C^k]] = 4\delta_{ij} \bar{Q}_C^k (\sigma_\mu)_{BA} P^\mu
\end{aligned} \tag{2.2}$$

and the following commutation relations with generators of the Poincaré group:

$$\begin{aligned}
&[J_{\mu\nu}, Q_A^j] = -\frac{1}{2i}\left(\sigma_{\mu\nu}\right)_{AB} Q_B^j \qquad [P_\mu, Q_A^j] = 0\\
&[J_{\mu\nu}, \bar{Q}_A^j] = -\frac{1}{2i}\left(\sigma_{\mu\nu}\right)^*_{AB} \bar{Q}_B^j \qquad [P_\mu, \bar{Q}_A^j] = 0.
\end{aligned} \tag{2.3}$$

Here σ_ν are the Pauli matrices, $\sigma_{\nu\sigma} = -\sigma_{\sigma\nu} = \sigma_\nu\sigma_\sigma$, $(.)_{AB}$ are the related matrix elements, and the asterisk denotes the complex conjugation.

It can easily be seen that the algebra $p(1,3;N)$ is a direct and natural generalization of the extended Poincaré superalgebra [15]. The last one also includes $10 + 4N$ elements which satisfy (2.1), (2.3), but instead of (2.2) the supercharges $Q_A^j,\ \bar{Q}_A^j$ fulfill the following anticommutation relations:

$$\begin{aligned}
&[Q_A^i, Q_B^j]_+ = 0 \qquad [\bar{Q}_A^i, \bar{Q}_B^j]_+ = 0\\
&[Q_A^i, \bar{Q}_B^j]_+ = 2\delta_{ij}(\sigma_\mu)_{AB} P^\mu
\end{aligned} \tag{2.4}$$

which, whenever satisfied, imply that relations (2.2) are also valid. However, the converse is not true.

Thus, the representations of the extended Poincaré *superalgebra* appear as particular representations of a more general algebraic structure—the extended Poincaré *parasuperalgebra* (cf the relations of the usual Fermi–Dirac or Bose–Einstein statistics with parastatistics [5]).

In the following sections we present the classification and explicit construction of IRs of the N-extended Poincaré parasuperalgebra defined by relations (2.1)–(2.3).

3. Classification of IRs

The IRs of the algebra p(1,3;N) can be specified by the eigenvalues of the appropriate Casimir operators.

First let us note that the Casimir operator of the Poincaré algebra $C_1 = P_\mu P^\mu$ commutes with all parasupercharges $Q_A^i,\ \bar{Q}_A^i$ so that it is also a Casimir operator for the Poincaré parasuperalgebra. The second less obvious, but nonetheless essential Casimir operator for the algebra $p(1, 3; N)$ can be obtained by extending the usual Pauli–Lubanski four-vector

$$W_\mu = \tfrac{1}{2}\varepsilon_{\mu\nu\rho\sigma} J^{\nu\rho} P^\sigma \tag{3.1}$$

to the following one:

$$B_\mu = W_\mu + X_\mu \tag{3.2}$$

where X_μ is defined by the bilinear combinations of parasupercharges:

$$\begin{aligned} X_0 &= \sum_{i=1}^N \frac{1}{8}\{[Q_1^i, \bar{Q}_1^i] + [Q_2^i, \bar{Q}_2^i]\} \qquad X_1 = \sum_{i=1}^N \frac{1}{8}\{[Q_1^i, \bar{Q}_2^i] + [Q_2^i, \bar{Q}_1^i]\} \\ X_2 &= \sum_{j=1}^N \frac{i}{8}\{[Q_2^j, \bar{Q}_1^j] + [\bar{Q}_2^j, Q_1^j]\} \qquad X_3 = \sum_{i=1}^N \frac{1}{8}\{[\bar{Q}_1^i, Q_1^i] + [Q_2^i, \bar{Q}_2^i]\}. \end{aligned} \tag{3.3}$$

As follows from (2.1) and (2.2), the four-vector B_μ satisfies the relations

$$[B_\mu, P_\nu] = 0 \qquad [B_\mu, J_{\nu\sigma}] = i(g_{\mu\nu}B_\sigma - g_{\mu\sigma}B_\nu) \tag{3.4a}$$

$$\begin{aligned} &[B_\mu, Q_A^i] = \tfrac{1}{2}P_\mu Q_A^i \qquad [B_\mu, \bar{Q}_A^i] = -\tfrac{1}{2}P_\mu \bar{Q}_A^i \\ &[B_\mu, B_\nu] = i\varepsilon_{\mu\nu\rho\sigma}P^\rho B^\sigma \end{aligned} \tag{3.4b}$$

from which we conclude that the operator

$$C_2 = P_\mu P^\mu B_\nu B^\nu - (B_\mu P^\mu)^2 \tag{3.5}$$

is the second Casimir operator of the algebra $p(1, 3; N)$.

We shall investigate the relations (2.2), (3.4*b*), which define an algebra of operators B_ν, Q_A^i and $\bar{Q}_A^i$ for any fixed set of eigenvalues p_ν of the operators P_ν.

As in the case of the ordinary Poincaré algebra [16] we shall classify IRs of the algebra $p(1, 3; n)$ according to the eigenvalues of C_1. We distinguish three classes of IRs, namely:

I. The time-like four-momentum case for which

$$P_\mu P^\mu = M^2 > 0. \tag{3.6a}$$

II. The light-like four-momentum case for which

$$P_\mu P^\mu = 0. \tag{3.6b}$$

III. The space-like four-momentum case for which

$$P_\mu P^\mu = -\eta^2 < 0. \tag{3.6c}$$

These will be studied separately in sections 4–7. The internal symmetries of the algebra $p(1, 3; n)$ and their representations will be introduced in section 8, and the physical relevance of the representations of $p(1, 3; N)$ will be discussed in section 9.

4. IRs of class I

For the time-like representations (3.6*a*) there exists an additional Casimir operator, namely $C_3 = P_0/|P_0|$, whose eigenvalues are $\varepsilon = \pm 1$. First we shall consider the case $\varepsilon = +1$ and determine 'a little Wigner parasuperalgebra' a_{I} associated with the time-like four-momentum taken in the form $P = (M, 0, 0, 0)$. For this particular choice of P we define the three-vector j_k by the indentities

$$B_k = W_k + X_k = -MS_k + X_k \equiv Mj_k \qquad k = 1, 2, 3 \tag{4.1}$$

and find that relations (3.4*b*) take the form

$$[B_0, Q^i_A] = \tfrac{1}{2} M\, Q^i_A \qquad [B_0, \bar{Q}^i_A] = -\tfrac{1}{2} M\, \bar{Q}^i_A \tag{4.2}$$

$$[j_k, Q^i_A] = [j_k, \bar{Q}^i_A] = 0. \tag{4.3}$$

Moreover, the vector j_k satisfies the commutation relations

$$[j_k, j_j] = i\varepsilon_{kjl} j_l \tag{4.4}$$

and the corresponding relations (2.2) reduce to

$$\begin{aligned}
&[Q^i_A, [\bar{Q}^j_B, Q^k_C]] = 4\delta_{ij}\delta_{AB} M Q^k_C \qquad [\bar{Q}^i_A, [Q^j_B, \bar{Q}^k_C]] = 4\delta_{ij}\delta_{AB} M \bar{Q}^k_C \\
&[Q^i_A, [Q^j_B, Q^k_C]] = [\bar{Q}^i_A, [\bar{Q}^j_B, \bar{Q}^k_C]] = 0.
\end{aligned} \tag{4.5}$$

Let us note two things: first, that according to (3.3), relations (4.2) become identities provided relations (4.5) are true, and secondly, that relations (4.5) are equivalent to relations (1.1) for para-fermions.

Thus we see that the little Wigner parasuperalgebra a_{I} is just a direct sum of the Lie algebra whose basis elements are j_a and of the algebra of operators Q_A, $\bar{Q}_A$ characterized by the double commutation relations (4.5).

Relations (4.4) define $so(3)$, i.e. the Lie algebra of the rotation group $SO(3)$. IRs of this algebra are well known and are determined by integers or half-integers j (see, e.g., [17]).

Relations (4.5) determine the algebra of the $2N$ creation and annihilation operators for para-fermions. These operators form a representation of the algebra $so(4N+1)$ [18].

To prove the isomorphism of the algebra (4.5) and $so(4N+1)$ explicitly, we express supercharges $Q^i_A, \bar{Q}^i_A$ and their commutators in terms of generators $S_{kl} = -S_{lk}$ $(k, l = 1, 2, \ldots, 4N+1)$ of $so(4N+1)$:

$$\begin{aligned}
Q^i_A &= -(-1)^A \sqrt{2M}(S_{4N+1\ 2N(A-1)+2i-1} - iS_{4N+1\ 2N(A-1)+2i}) \\
\bar{Q}^i_A &= -(-1)^A \sqrt{2M}(S_{4N+1\ 2N(A-1)+2i-1} + iS_{4N+1\ 2N(A-1)+2i})
\end{aligned} \tag{4.6}$$

$$\begin{aligned}
[Q^i_A, \bar{Q}^k_B] = (-1)^{(A+B)} 2M \big[&-S_{2N(A-1)+2i-1\ 2N(B-1)+2k} - S_{2N(B-1)+2k-1\ 2N(A-1)+2i} \\
&+ i\left(S_{2N(A-1)+2i-1\ 2N(B-1)+2k-1} + S_{2N(A-1)+2i\ 2N(B-1)+2k}\right)\big] \\
[Q^i_A, Q^k_B] = (-1)^{(A+B)} 2M \big[&S_{2N(A-1)+2i-1\ 2N(B-1)+2k} + S_{2N(A-1)+2i\ 2N(B-1)+2k-1} \\
&+ i\left(S_{2N(A-1)+2i-1\ 2N(B-1)+2k-1} - S_{2N(A-1)+2i\ 2N(B-1)+2k}\right)\big] \\
[\bar{Q}^i_A, \bar{Q}^k_B] = (-1)^{(A+B)} 2M \big[&-S_{2N(A-1)+2i-1\ 2N(B-1)+2k} - S_{2N(A-1)+2i\ 2N(B-1)+2k-1} \\
&+ i\left(S_{2N(A-1)+2i-1\ 2N(B-1)+2k-1} - S_{2N(A-1)+2i\ 2N(B-1)+2k}\right)\big].
\end{aligned} \tag{4.7}$$

Equations (4.6), (4.7) are invertible, so that

$$\begin{aligned}
S_{4N+1\ 2N(A-1)+2i-1} &= -(-1)^A \frac{1}{2\sqrt{2M}} \left(\bar{Q}^i_A + Q^i_A\right) \\
S_{4N+1\ 2N(A-1)+2i} &= (-1)^A \frac{i}{2\sqrt{2M}} \left(\bar{Q}^i_A - Q^i_A\right) \\
S_{2N(A-1)+2i-1\ 2N(B-1)+2k} &= -(-1)^{(A+B)} \frac{1}{8M} \left[\bar{Q}^i_A + Q^i_A,\ \bar{Q}^k_B - Q^k_B\right] \\
S_{2N(A-1)+2i-1\ 2N(B-1)+2k-1} &= -(-1)^{(A+B)} \frac{i}{8M} \left[\bar{Q}^i_A - Q^i_A,\ \bar{Q}^k_B + Q^k_B\right] \\
S_{2N(A-1)+2i\ 2N(B-1)+2k} &= (-1)^{(A+B)} \frac{i}{8M} \left[\bar{Q}^i_A - Q^i_A,\ \bar{Q}^k_B - Q^k_B\right].
\end{aligned} \tag{4.8}$$

Using (4.5), (4.8) we obtain the following commutation relations:

$$[S_{kl}, S_{mn}] = -i(g_{km}S_{ln} + g_{ln}S_{km} - g_{kn}S_{lm} - g_{lm}S_{kn}) \tag{4.9}$$

(where $g_{kl} = -\delta_{kl}$, in which δ_{kl} is the Kronecker symbol) which are specific to the Lie algebra $so(4N+1)$.

The IRs of the algebra $so(4N+1)$ are determined by the set of numbers $(n_1, n_2, \ldots, n_{2N})$ which are either integers or half-integers and satisfy the inequalities $n_1 \geqslant n_2 \cdots \geqslant n_{2N} \geqslant 0$. For the explicit form of the matrices S_{kl} see, e.g., [18].

Thus we have proved that for $P_\nu P^\nu > 0$ the algebra a_{I} is equivalent to a direct sum of the algebras $so(3)$ and $so(4N+1)$:

$$a_{\mathrm{I}} = so(3) \oplus so(4N+1). \tag{4.10}$$

As follows from the above discussion, the IRs of the algebra $p(1,3;n)$ that belong to class I with positive energy are labelled by the following sets of numbers: $(M, j, \varepsilon = 1, n_1, n_2, \ldots, n_{2N})$. The explicit expressions for the corresponding Pauli–Lubanski vector and for the parasupercharges can be found from (3.2), (4.1), (4.6) by means of the Lorentz transformation (specified in the appendix) and are of the form:

$$W_0 = p_a S_a \qquad W_a = \varepsilon M S_a + \frac{p_a S_b p_b}{(E+M)} \tag{4.11}$$

$$\begin{aligned} Q_1^i &= \frac{1}{\sqrt{E+M}}[(S_{4N+1\ 2i-1} - iS_{4N+1\ 2i})(E+M+\varepsilon p_3) \\ &\quad - \varepsilon(S_{4N+1\ 2N+2i-1} - iS_{4N+1\ 2N+2i})(p_1 - ip_2)] \\ Q_2^i &= \frac{1}{\sqrt{E+M}}[\varepsilon(S_{4N+1\ 2i-1} - iS_{4N+1\ 2i})(p_1 + ip_2) \\ &\quad - (S_{4N+1\ 2N+2i-1} - iS_{4N+1\ 2N+2i})(E+M-\varepsilon p_3)] \\ \bar{Q}_A &= Q_A^+ \end{aligned} \tag{4.12}$$

where

$$\varepsilon = 1 \qquad E = \sqrt{M^2 + p^2} \qquad p^2 = p_1^2 + p_2^2 + p_3^2$$

$$S_a = \frac{1}{2}\sum_{i=0}^{n-1}\left(\frac{1}{2}\varepsilon_{a\ b+4i\ c+4i}S_{b+4\ c+4i} + S_{4(i+1)\ a+4i}\right) \tag{4.13}$$

and ε_{abc} is an absolutely antisymmetric tensor of rank 3. The explicit expressions for the generators of the Poincaré group, corresponding to the Pauli–Lubanski vector (4.11), are given by

$$\begin{aligned} P_0 &= \varepsilon E \qquad P_a = p_a \\ J_{ab} &= x_a p_b - x_b p_a + \varepsilon_{abc} S_c \\ J_{0a} &= x_0 p_a - \frac{i\varepsilon}{2}\left[\frac{\partial}{\partial p_a}, E\right]_+ - \varepsilon\frac{\varepsilon_{abc} p_b S_c}{E+M} \end{aligned} \tag{4.14}$$

where $x_a = i\partial/\partial p_a$, and x_0 is a parameter which can be set zero without loss of generality.

Let us remark that the operators (4.12)–(4.14) are hermitian with respect to the following scalar product:

$$(\psi_1, \psi_2) = \int_{\infty}^{\infty} \psi_1^+ \psi_2 \, \mathrm{d}^3 p \tag{4.15}$$

where $\psi_\alpha = \psi_\alpha(\mathbf{p})$ are m-component wavefunctions, forming a basis of the m-dimensional IR $D(n_1, n_2, \ldots, n_{2N})$ of the algebra $so(4N+1)$.

In contrast to the case of Poincaré superalgebra [7] for which the energy sign operator has positive eigenvalues only, the algebra $p(1,3;N)$ admits a representation with both signs of the Casimir operator C_3.

For the case $\varepsilon = -1$, i.e. for negative energy, the rest frame four-momentum is of the form $P = (-M, 0, 0, 0)$ with $M > 0$. In this frame relations (2.2) are of the form

$$\begin{aligned} &[Q^i_A, [\bar{Q}^j_B, Q^k_C]] = -4\delta_{ij}\delta_{AB} M Q^k_C \qquad [\bar{Q}^i_A, [Q^j_B, \bar{Q}^k_C]] = -4\delta_{ij}\delta_{AB} M \bar{Q}^k_C \\ &[Q^i_A, [Q^j_B, Q^k_C]] = [\bar{Q}^i_A, [\bar{Q}^j_B, \bar{Q}^k_C]] = 0 \end{aligned} \tag{4.16}$$

(cf equations (4.5)).

Similarly to (4.6)–(4.8), it is possible to show that the algebra (4.16) is isomorphic to the algebra $so(1, 4N)$ whose representations are discussed in [19]. The corresponding basis elements of the algebra $p(1,3;N)$ again have the form (4.12)–(4.14) where, however, $\varepsilon = -1$ and S_{kl} are now the basis elements of an IR of the algebra $so(1, 4N)$ which satisfy the commutation relations (4.9) (in which the non-zero components of $g_{\mu\nu}$ are $g_{\nu\nu} = -1$, $\nu \neq 4N+1$ and $g_{4N+1\,4N+1} = 1$). A description of the IRs of the algebra $so(1, 4N)$ in complete detail can be found in the book [19].

Thus, in this section we have enumerated all the class I IRs of the extended Poincaré parasuperalgebra $p(1,2;N)$ and have found the explicit expressions of the basis elements.

5. IRs of class II

To this class of IRs we have again the additional Casimir operator $C_3 = P_0/|P_0|$ with the eigenvalues $\varepsilon = \pm 1$. As previously, we shall consider the case $\varepsilon = +1$ first.

To determine the corresponding little Wigner parasuperalgebra a_{II} we choose the light-like four-momentum P in the form $P = (M, 0, 0, M)$. Then the algebra (2.2) reduces to the form

$$\begin{aligned} &[Q^i_2, [\bar{Q}^k_2, Q^j_2]] = 8M\delta_{ik} Q^j_2 \qquad [\bar{Q}^i_2, [Q^k_2, \bar{Q}^j_2]] = 8M\delta_{ik}\bar{Q}^j_2 \\ &[Q^i_2, [Q^k_2, Q^j_2]] = [\bar{Q}^i_2, [\bar{Q}^k_2, \bar{Q}^j_2]] = 0 \end{aligned} \tag{5.1}$$

$$\begin{aligned} &[Q^i_2, [\bar{Q}^k_2, Q^j_1]] = 8M\delta_{ik} Q^j_1 \qquad [\bar{Q}^i_2, [Q^k_2, \bar{Q}^j_1]] = 8M\delta_{ik}\delta_{AB}\bar{Q}^j_1 \\ &[Q^i_1, [\bar{Q}^k_1, Q^j_A]] = [\bar{Q}^i_1, [Q^k_1, Q^j_A]] = 0 \\ &[Q^i_A, [Q^k_B, Q^i_C]] = [\bar{Q}^i_A, [\bar{Q}^k_B, \bar{Q}^j_C]] = 0. \end{aligned} \tag{5.2}$$

Now expressing Q^j_A in terms of S_{kl} $(k, l = 1, 2, \ldots, 2N+1)$, namely

$$\begin{aligned} &Q^j_2 = 2\sqrt{M}\left(S_{2N+1\,2j} + iS_{2N+1\,2j-1}\right) \\ &\bar{Q}^j_2 = 2\sqrt{M}\left(S_{2N+1\,2j} - iS_{2N+1\,2j-1}\right) \\ &\left[\bar{Q}^k_2, Q^j_2\right] = 4M\left(iS_{2k\,2j} + iS_{2k-1\,2j-1} + S_{2k-1\,2j} - S_{2k\,2j-1}\right). \end{aligned} \tag{5.3}$$

we find that operators S_{kl} satisfy relations (4.9) with $g_{kl} = -\delta_{kl}$, i.e. they form a basis of the algebra $so(2N+1)$. Since relations (5.3) are invertible

$$
\begin{aligned}
S_{2N+1\,2j} &= \frac{1}{4\sqrt{M}}\big(Q_2^j + \bar{Q}_2^j\big) \\
S_{2N+1\,2j-1} &= \frac{i}{4\sqrt{M}}\big(-Q_2^j + \bar{Q}_2^j\big) \\
S_{2j\;2k} &= -\frac{i}{16M}\big[Q_2^j + \bar{Q}_2^j, Q_2^k + \bar{Q}_2^k\big] \\
S_{2j-1\;2k-1} &= \frac{i}{16M}\big[Q_2^j - \bar{Q}_2^j, Q_2^k - \bar{Q}_2^k\big] \\
S_{2j\;2k-1} &= -\frac{1}{16M}\big[Q_2^j + \bar{Q}_2^j, Q_2^k - \bar{Q}_2^k\big]
\end{aligned}
\tag{5.4}
$$

the algebra (5.1) reduces to the algebra $so(2N+1)$ whose IRs are labelled by the set of N numbers which are all integers or half-integers and which satisfy the inequalities $n_1 \geqslant n_2 \geqslant \cdots \geqslant n_N \geqslant 0$.

In order to describe the structure of the little Wigner parasuperalgebra a_{II} we notice two facts. First, the relations (5.2) only have trivial solutions for Q_1^j and $\bar{Q}_1^j$.

Secondly, in accordance with (3.4*b*), (5.4), (4.4) we obtain

$$B_3 = B_0 \qquad [B_0, B_1] = iMB_2 \qquad [B_0, B_2] = -iMB_1 \qquad [B_0, B_1] = 0. \tag{5.5}$$

Defining the operators $T_0,\ T_1,\ T_2$ by the following formulae:

$$B_0 = W_0 + X_0 \equiv M(T_0 - \tfrac{1}{2}(Nn_1 - \hat{S}_3)) \qquad B_1 = W_1 \equiv T_1 \qquad B_2 = W_2 \equiv T_2 \tag{5.6a}$$

where n_1 is the main quantum number characterizing the IR of the algebra $so(2N+1)$,

$$\hat{S}_3 = S_{12} + S_{34} + \cdots + S_{2N-1\,2N} \tag{5.6b}$$

we obtain the following relations from (5.5):

$$[T_0, T_1] = iT_2 \qquad [T_0, T_2] = -iT_1 \qquad [T_1, T_2] = 0 \tag{5.7}$$

$$[T_0, S_{ab}] = [T_1, S_{ab}] = [T_2, S_{ab}] = 0. \tag{5.8}$$

We conclude from (5.8) that the algebra a_{II} for class II representations is a direct sum of the algebras $so(2N+1)$ and $e(2)$, which are determined by relations (4.4) and (5.7), respectively.

Thus, we have found the explicit form of the operators $W_\nu,\ Q_A^i,\ \bar{Q}_A^i$ in the reference frame $P = (M, 0, 0, M)$. To obtain explicit expressions for these operators (and the corresponding generators $P_\nu,\ J_{\nu\sigma}$) in an arbitrary reference frame it is sufficient to make the corresponding rotation. As a result we get

$$
\begin{aligned}
Q_1^i &= \frac{\sqrt{2}(-p_1 + ip_2)}{\sqrt{p+p_3}}(S_{2N+1\,2i} - iS_{2N+1\,2i-1}) \\
\bar{Q}_1^i &= \frac{\sqrt{2}(-p_1 - ip_2)}{\sqrt{p+p_3}}(S_{2N+1\,2i} + iS_{2N+1\,2i-1}) \\
Q_2^i &= \sqrt{2(p+p_3)}(S_{2N+1\,2i} - iS_{2N+1\,2i-1}) \\
\bar{Q}_2^i &= \sqrt{2(p+p_3)}(S_{2N+1\,2i} + iS_{2N+1\,2i-1})
\end{aligned}
\tag{5.9}
$$

and

$$
\begin{aligned}
P_0 &= \varepsilon p \qquad P_a = p_a \\
J_{ab} &= x_a p_b - x_b p_a + \varepsilon_{abc}\hat{T}_0 \frac{p_c + \delta_{c3} p}{p + p_3} \\
J_{0a} &= x_0 p_a - \frac{1}{2}\varepsilon[p, x_a]_+ + \frac{\varepsilon_{abc} T_b p_c}{p^2} - \frac{\varepsilon_{abc} p_b n_c (\varepsilon \hat{T}_0 p^2 - T_a p_a)}{p^2(p+p_3)}
\end{aligned}
\tag{5.10}
$$

where

$$p = \sqrt{p_1^2 + p_2^2 + p_3^2} \qquad n_1 = n_2 = 0 \qquad n_3 = 1$$
$$T_3 = 0 \qquad \hat{T}_0 = T_0 - \tfrac{1}{2}(Nn_1 + \hat{S}_3)$$

$\hat{S}_3$ is defined in (5.6*b*), and T_0, T_1, T_2 are the basis elements of the IRs of the algebra $e(2)$ specified in (5.7).

In the important case $T_1^2 + T_2^2 = 0$ (i.e. for representations with a discrete spin), equations (5.10) are simplified and reduced to the form

$$P_0 = \varepsilon p \qquad P_a = p_a$$
$$J_{ab} = x_a p_b - x_b p_a + \varepsilon_{abc}(2\lambda - Nn_1 - \hat{S}_3)\frac{p_c + \delta_{c3} p}{p + p_3} \tag{5.11}$$
$$J_{0a} = x_0 p_a - \frac{1}{2}\varepsilon[p, x_a]_+ \frac{\varepsilon_{abc} p_b n_c (2\lambda - Nn_1 - \hat{S}_3)}{p^2(p + p_3)}$$

where both λ and $n_1 > 0$ are either arbitrary integers or half-integers.

For $\varepsilon = -1$ we can choose the light-like four-momentum in the form $P = (-M, 0, 0, -M)$. The relations corresponding to (2.2) can be obtained from (5.1), (5.2) by taking $M \to -M$. Then, using expression (5.3) again, we conclude that S_{kl} satisfy relations (4.9) where the non-zero components of $g_{\mu\nu}$ are $g_{nn} = -1, n < 2N + 1$, and $g_{2N+1/2N+1} = 1$. In other words, S_{kl} form just the algebra $so(1, 2N)$, and the algebra a_{II} is a direct sum $e(2) \oplus so(1, 2N)$.

Consequently, the above IRs of the extended Poincaré parasuperalgebra for the light-like four-momenta are qualitatively different for $\varepsilon = +1$ and $\varepsilon = -1$. In the case $\varepsilon = +1$ these IRs are labelled by $N + 2$ quantum numbers $\varepsilon = 1, r, n_1, n_2, \ldots, n_N$ (where $n_1, n_2, \ldots, n_N$ are Gelfand–Zetlin numbers for $so(2N + 1)$, and r is an eigenvalue of the Casimir operator $T_1^2 + T_2^2$ of the algebra $e(2)$), or for $r = 0$ by ($\varepsilon = 1, \lambda, n_1, n_2, \ldots, n_N$) (where λ are integers or half-integers). For $\varepsilon = -1$ the IRs are specified by the eigenvalues of the Casimir operators of the non-compact algebras $e(2)$ and $so(1, 2N)$, which are described, e.g., in [15] and [19].

6. IRs of class III

To obtain the corresponding little Wigner parasuperalgebra a_{III} we choose the space-like four-momentum in the form $P = (0, 0, 0, \eta)$. The corresponding double commutation relations (2.2) reduce to the form

$$[Q_A^i, [\bar{Q}_B^j, Q_C^k]] = (-1)^A \delta_{ij}\delta_{AB} 4\eta Q_C^k \qquad [\bar{Q}_A^i, [Q_B^j, \bar{Q}_C^k]] = (-1)^A \delta_{ij}\delta_{AB} 4\eta \bar{Q}_C^k$$
$$[Q_A^i, [Q_B^j, Q_C^k]] = [\bar{Q}_A^i, [\bar{Q}_B^j, \bar{Q}_C^k]] = 0. \tag{6.1}$$

Introducing the operators $\tilde{J}_{01}$, $\tilde{J}_{02}$ and $\tilde{J}_{12}$ in accordance with the following relations:

$$\eta \tilde{J}_{01} \equiv B_2 = J_{01}\eta + X_2 \qquad \eta \tilde{J}_{12} \equiv B_1 = -J_{02}\eta + X_1$$
$$\eta \tilde{J}_{02} \equiv B_0 = -J_{12}\eta + X_0 \tag{6.2}$$

and taking into account that $B_3 = X_3$, from equation (3.4*b*) we find that

$$[\tilde{J}_{\alpha\beta}, Q_A] = [\tilde{J}_{\alpha\beta}, \bar{Q}_A] = 0 \qquad \alpha, \beta = 0, 1, 2 \tag{6.3}$$
$$[\tilde{J}_{\alpha\beta}, \tilde{J}_{\rho\sigma}] = i(g_{\alpha\sigma}\tilde{J}_{\beta\rho} + g_{\beta\rho}\tilde{J}_{\alpha\sigma} - g_{\alpha\rho}\tilde{J}_{\beta\sigma} - g_{\beta\sigma}\tilde{J}_{\alpha\rho}) \tag{6.4}$$

where $g_{00} = -g_{11} = -g_{22} = 1$, $g_{\alpha\beta} = 0$, $\alpha \neq \beta$.

In accordance with (6.1)–(6.4) the algebra $a_{\rm III}$ corresponding to space-like momenta is equivalent to the direct sum of the algebra $so(1,2)$ (defined by relations (6.4)) and the algebra defined by the double commutation relations (6.1). Expressing the parasupercharges in terms of S_{kl} via (4.6), (4.7) with M replaced by η, we conclude that the corresponding matrices S_{kl} satisfy relations (4.9), where the non-zero components of $g_{\mu\nu}$ are $g_{nn} = 1, n < 2N+1$ and $g_{nn} = -1, n \geqslant 2N$. In other words, relations (6.1) specify the algebra $so(2N, 2N+1)$.

Thus we have shown that the little Wigner parasuperalgebra $a_{\rm III}$ for representations of class III is isomorphic to a direct sum of two non-compact algebras, namely, of $so(1,2)$ and $so(2N, 2N+1)$.

Taking the generators $P_\mu, J_{\mu\nu}, Q_A^i$ and $\bar{Q}_A^i$ in the form expressed in (6.2), (4.6), (3.1) and performing Lorentz transformation and rotation corresponding to their transition to an arbitrary reference frame (see equations (A.2), (A.3)), we find that the basis elements of the extended Poincaré parasuperalgebra can be written as

$$
\begin{aligned}
&P_\mu = p_\mu \qquad J_{ab} = x_a p_b - x_b p_a + \tilde{S}_{ab} \\
&J_{0a} = x_0 p_a - \tfrac{1}{2}[x_a, p_0]_+ + \tilde{S}_{0a} \\
&J_{a3} = x_a p_3 - x_3 p_a - \frac{\tilde{S}_{ab} p_b - \tilde{S}_{a0} p_0}{p_3 + \eta} \\
&J_{03} = x_0 p_3 - \frac{1}{2}[x_3, p_0]_+ - \frac{\tilde{S}_{0a} p_a}{p_3 + \eta} \\
&Q_1^i = \frac{1}{\sqrt{\eta + p_3}} \big[(S_{4N+1\ 2i-1} - iS_{4N+\ 2i})(\eta + p_3 - p_0) \\
&\qquad + (S_{4N+1\ 2N+2i-1} - iS_{4N+1\ 2N+2i})(p_1 - ip_2)\big] \\
&Q_2^i = \frac{1}{\sqrt{\eta + p_3}} \big[(S_{4N+1\ 2i-1} - iS_{4N+1\ 2i})(p_1 + ip_2) \\
&\qquad - (S_{4N+1\ 2N+2i-1} - iS_{4N+1\ 2N+2i})(\eta + p_3 + p_0)\big] \\
&\bar{Q}_A = Q_A^+
\end{aligned}
\tag{6.5}
$$

where

$$
p_0^2 = \mathbf{p}^2 - \eta^2 \qquad \tilde{S}_{12} = \tilde{J}_{12} + S_3
$$
$$
\tilde{S}_{01} = \tilde{J}_{01} + S_1 \qquad \tilde{S}_{02} = \tilde{J}_{02} + S_2.
$$

Here the $\tilde{J}_{\alpha\beta}$ are the basis elements of the algebra $so(1,2)$ introduced in (6.4), and the S_a are defined in (4.13) with S_{kl} being the elements of the algebra $so(2N+1, 2N)$.

7. Special representations

In this section we shall show how to construct special representations of the algebra $p(1,3;N)$ namely representations in which the Poincaré generators $P_\mu, J_{\mu\nu}$ have the form

$$
P_\mu = i\frac{\partial}{x_\mu} \qquad J_{\mu\nu} = x_\mu \frac{\partial}{x_\nu} - x_\nu \frac{\partial}{x_\mu} + S_{\mu\nu} \tag{7.1}
$$

with $S_{\nu\sigma}$ being numerical matrices. These representations (in which the spin part $S_{\mu\nu}$ of any generator $J_{\mu\nu}$ commutes with the orbital part $x_\mu\, \partial/x_\nu - x_\nu\, \partial/x_\mu$) are frequently used in various physical applications.

We take $S_{\nu\sigma}$ in the form

$$S_{ab} = \varepsilon_{abc} S_c \qquad S_{0a} = i S_a \tag{7.2}$$

where the S_a are the matrices defined in (4.13). Then the corresponding parasupercharges can be expressed as

$$\begin{aligned}
Q_1^j &= \sqrt{2M}(S_{4N+1\ 2j-1} - iS_{4N+1\ 2j}) \\
Q_2^j &= -\sqrt{2M}[(S_{4N+1\ 2N+2j-1} - iS_{4N+1\ 2N+2j}) \\
\bar{Q}_1 &= \sqrt{\frac{2}{M}}\big[(p_3 - p_0)(S_{4N+1\ 2j-1} + iS_{4N+1\ 2j}) \\
&\qquad + (p_1 + ip_2)(S_{4N+1\ 2N+2j-1} + iS_{4N+1\ 2N+2j})\big] \\
\bar{Q}_2 &= -\sqrt{\frac{2}{M}}\big[(p_0 + p_3)(S_{4N+1\ 4N+2j-1} + iS_{4N+1\ 4N+2j}) \\
&\qquad + (p_1 - ip_2)(S_{4N+1\ 2j-1} + iS_{4n+1\ 2j})\big].
\end{aligned} \tag{7.3}$$

It is easy to verify that the operators (7.1)–(7.3) satisfy relations (2.1), (2.2), (2.4), and consequently realize a representation of the extended Poincaré parasuperalgebra. Moreover, assuming that $P_\nu P^\nu = M^2 > 0,\ p_0 = E = (p^2 + M^2)^{1/2}$, it is possible to prove that this representation is irreducible. Indeed, the corresponding generators (7.2), (7.3) can be reduced to the form (4.12), (4.14) using the transformation

$$\begin{aligned}
J_{\mu\nu} &\to U J_{\mu\nu} U^{-1} \qquad & P_\mu &\to U P_\mu U^{-1} \\
Q_A &\to U Q_A U^{-1} \qquad & \bar{Q}_A &\to U \bar{Q}_A U^{-1}
\end{aligned} \tag{7.4}$$

where

$$U = \exp\left(\frac{iS_{0a}p_a}{p} \operatorname{arctanh} \frac{p}{E}\right). \tag{7.5}$$

We notice that supercharges Q_A^j and $\bar{Q}_A^j$ are not conjugated with respect to the usual scalar product (4.15). However, they are conjugated with respect to the following scalar product:

$$(\psi_1, \psi_2) = \int \psi_1 M \psi_2 \, \mathrm{d}^3 x$$

in which $M = U^+ U = \exp\left(\frac{2iS_{0a}p_a}{p} \operatorname{arctanh} \frac{p}{E}\right)$ is a positive definite metric operator.

8. Internal symmetries

Now shall demonstrate that any IR of the algebra $p(1, 3; N)$ described in the previous sections, can be extended to a representation of the algebra including $p(1, 3; N)$ and the internal symmetry algebra. In other words, the carier space of an IR of the extended Poincaré parasuperalgebra $p(1, 2; N)$, appears to be also a carier space of a representation of the algebra $u(N)$ whose the N^2, generators $T_{ab},\ a, b = 1, 2, \ldots, N$, satisfy the following relations:

$$\left[P_\mu, T_{ab}\right] = \left[J_{\mu\nu}, T_{ab}\right] = 0 \tag{8.1a}$$

$$\left[T_{ab}, Q_A^i\right] = f_{ab}^{ik} Q_A^k \tag{8.1b}$$

$$[T_{ab}, T_{cd}] = \delta_{bc} T_{ab} - \delta_{ad} T_{bc}. \tag{8.1c}$$

The last relation simply determines the algebra $u(N)$ in the Okubo basis [20].

Let us start with the case of light-like four-momenta considered in section 5. The corresponding parasupercharges are represented in terms of matrices S_{kl} satisfying the algebra $so(2N+1)$. The related generators of the internal symmetry algebra can be expressed in terms of basis elements of the algebra $so(2N+1)$ as follows:

$$
\begin{aligned}
T_{nn} &= \frac{1}{N}\left(1 - \hat{S}_3 + N S_{2n-1\,2n}\right)\\
T_{ab} &= \tfrac{1}{2}\left(S_{2a\,2b-1} - S_{2a-1\,2b} + iS_{2b-1\,2a-1} - iS_{2a\,2b}\right) \qquad b > a \qquad (8.2)\\
T_{ba} &= T_{ab}^{+}
\end{aligned}
$$

where $\hat{S}_3$ is the matrix defined in (5.6*b*), $a, b, = 1, 2, \ldots, N$.

It is easy to verify that the operators (8.2) satisfy relations (8.1), where

$$
f_{ab}^{ik} = \begin{cases} \delta_{in}\delta_{ka} - \frac{1}{N}\delta_{ik} & a = b = n\\ 2\delta_{ia}\delta_{kb} & b > a\\ 2\delta_{ib}\delta_{ka} & b < a. \end{cases} \qquad (8.3)
$$

For the case $P_\mu P^\mu > 0$ the parasupercharges can be expressed in terms of the basis elements of the algebra $so(4N+1)$, and the generators of the internal symmetry algebra can be chosen in the form

$$
\begin{aligned}
T_{nn} &= \frac{1}{N}\left(1 - \Lambda + N S_{2n-1\,2n} + N S_{2N+2n-1 2N+2n}\right)\\
T_{ab} &= \tfrac{1}{2}\left(S_{2a\,2b-1} - S_{2a-1\,2b} + S_{2N+2a\,2N+2b-1} - S_{2N+2a-1\,2N+2b} + iS_{2b-1\,2a-1}\right.\\
&\qquad \left. - iS_{2a\,2b} + iS_{2N+2b-1\,2N+2a-1} - iS_{2N+2a\,2N+2b}\right) \qquad a < b \qquad (8.4)\\
T_{ba} &= T_{ab}^{+}
\end{aligned}
$$

where

$$
\Lambda = S_{12} + S_{34} + S_{56} + \cdots + S_{4N-1\,4N} \qquad a, b, = 1, 2, \ldots, N.
$$

The commutation relations of T_{ab} with $P_\mu,\ J_{\mu\nu},\ Q_A^i$ are given in (8.1), (8.3).

For class III IRs the generators of the internal symmetry group again take the form (8.4) where, however, the S_{kl} are the basis elements of the Lie algebra of the non-compact group $SO(2N, 4N+1)$.

Thus, as in the case of the Poincaré superalgebra (cf [2, 3]), the algebra $p(1, 3; N)$ can be extended by the generators of the internal symmetry group which is then quite simply the familiar group $U(N)$. These generators can be expressed in terms of linear combinations of the basis elements of the orthogonal algebras which generate the IRs of the extended Poincaré parasuperalgebra.

9. Discussion

In the previous sections we have classified and constructed the irreducible representations (IRs) of the Poincaré parasuperalgebra with an arbitrary number of parasupercharges. Moreover, we have presented the explicit form of the basis elements of the algebra $p(1, 3; N)$ in terms of the matrices belonging to the IRs of the (pseudo)orthogonal algebras $so(4N+1)$, $so(1, 4N)$, ..., etc. In other words, we had taken an alternative route to the usual paragrasmanian variables [5].

Let us briefly discuss the spin content of the corresponding parasupermultiplets and possible physical interpretations of the representations obtained.

We start with IRs of class I. These representations are reducible with respect to the Poincaré algebra $p(1,3)$, which is a subalgebra of $p(1,3;N)$.

Let us consider the case when the internal spin j is equal to zero. Starting with (4.14) and calculating the corresponding Casimir operator $C = W_\nu W^\nu$ of the subalgebra $p(1,3)$, we obtain

$$W_\mu W^\mu = M^2\mathbf{S}^2 \tag{9.1}$$

where $\mathbf{S} = (S_1, S_2, S_3)$ and the S_a are the matrices defined in (4.13). These matrices form a subalgebra $so(3)$ of the algebra $so(4N+1)$ and realize a reducible representation of this subalgebra. Reducing the IR $D(n_1, n_2, \ldots, n_{2N})$ to the corresponding representations of the spin algebra $so(3)$, we obtain the following set of eigenvalues of the Casimir operator of the Poincaré group:

$$\begin{aligned} &W_\mu W^\mu \psi = -M^2 s(s+1)\psi \\ &s = N\frac{n_1+n_2}{2},\ N\frac{n_1+n_2-1}{2},\ N\frac{n_1+n_2-2}{2},\ \ldots,\ 0 \end{aligned} \tag{9.2}$$

where n_1 and n_2 are respectively the first and second quantum numbers enumerating the corresponding IR of the algebra $so(4N+1)$.

For the case $j \neq 0$ (see equations (4.7), (4.14)) the possible spin values can be found as a result of summation of the two momenta, i.e. j and S of (4.13). Instead of (9.2) we get

$$\begin{aligned} &s = N\frac{n_1+n_2}{2} + j,\ N\frac{n_1+n_2}{2} + j - 1,\ \ldots,\ s_0 \\ &s_0 = \begin{cases} 0 & \dfrac{N(n_1+n_2)}{2} \geqslant j \\ j - \dfrac{n_1+n_2}{2} & \dfrac{N(n_1+n_2)}{2} < j. \end{cases} \end{aligned} \tag{9.3}$$

Thus the IRs of the algebra $p(1,3;N)$ can be viewed as being in correspondence with the parasupermultiplets of particles with spins given by equations (9.2), (9.3).

As in the case of the Poincaré superalgebra [2], the parasupermultiplets contain bosons as well as fermions.

Let us now consider one example of an IR of $p(1,3;N)$, namely, for $n_1 = n_2 = \cdots = n_{2N} = 1/2$. It appears that these representations are IRs of the Poincaré superalgebra, since in this case the corresponding operators Q_A and $\bar{Q}_A$ satisfy the usual anticommutation relations (2.4) for supercharges.

Thus, we had obtained the IRs of the Poincaré superalgebra as a particular (and the simplest) case of the representations of our more general algebra $p(1,3;N)$.

Now consider the IRs of class II with discrete spins. The corresponding basis elements are defined in (5.9), (5.11).

The related IRs of the algebra $p(1,3;n)$ are reducible with respect to the subalgebra $p(1,3)$. This can be seen by calculating the additional Casimir operator C of the $p(1,3)$ in these representations. We obtain

$$C = \frac{J_{12}p_3 + J_{31}p_2 + J_{23}p_1}{p} = \lambda - \frac{1}{2}Nn_1 - \frac{1}{2}\hat{S}_3$$

and its eigenvalues $\bar{\lambda}$ (associated with helicities of particles) in the form

$$\bar{\lambda} = \lambda,\ \lambda - \tfrac{1}{2},\ \lambda - 1,\ \ldots,\ \lambda - Nn_1. \tag{9.4}$$

Thus, the corresponding parasupermultiplets include both bosons and fermions, whose helicities are given in (9.4).

For $n_1 = n_2 = \cdots = n_N = 1/2$ these IRs of $p(1, 3; N)$ again reduce to the IRs of the Poincaré superalgebra.

In conclusion, we notice that in the present paper we have studied the simplest 'paraextension' of the Poincaré algebra, which includes only para-fermionic charges. The other extensions of the Poincaré algebra, including central parasupercharges and also para-fermionic and para-bosonic charges, will be considered elsewere.

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Appendix. Lorentz transformations

Here we present the explicit Lorentz transformations of four-vectors and spinors which have been used in body of the paper.

The transformation $p'_\mu \to p_\mu = A_{\mu\nu} p'^\nu$, where p_μ are components of a time-like four-momenta $P = (p_0, p_1, p_2, p_3)$, and p'_μ are the related components in the c.m. frame, is performed by the matrix

$$A = \exp\left(-\frac{iS_{0a}p_a}{p}\,\mathrm{arctanh}\,\frac{p}{E}\right) = 1 - \frac{i}{M}S_{0a}p_a - \frac{1}{M(M+E)}\,(S_{0a}p_a)^2 \qquad \text{(A.1)}$$

where $p = \sqrt{p_1^2 + p_2^2 + p_3^2}$, $E = \sqrt{p^2 + M^2}$, and the matrices S_{0a} have the following non-zero elements:

$$(S_{01})_{12} = (S_{01})_{21} = (S_{02})_{13} = (S_{02})_{31} = (S_{03})_{14} = (S_{03})_{41} = i.$$

The corresponding transformation of the Weyl spinors is $Q_A \to B_{AB}Q_B$, where the B_{AB} are the elements of the following matrix:

$$B = \exp\left(\frac{\sigma_a p_a}{2p}\,\mathrm{arctanh}\,\frac{p}{E}\right) = \frac{E + M + \sigma_a p_a}{\sqrt{2M(E+M)}}$$

in which σ_a are the Pauli matrices.

The rotation transformation $(M, 0, 0, M) \to (p_0, p_1, p_2, p_3)$ for a light-like four-momentum is performed by the matrix

$$A = \exp\left(-\frac{iS_{3a}p_a}{\sqrt{p_1^2+p_2^2}}\,\arctan\frac{\sqrt{p_1^2+p_2^2}}{p_3}\right) = 1 - i\frac{S_{3a}p_a}{p} - \frac{(S_{3a}p_a)^2}{p(p+p_3)}$$

in which the non-zero elements of $S_{3a} (a = 1, 2)$ are given by

$$(S_{31})_{24} = -(S_{31})_{42} = (S_{32})_{34} = -(S_{32})_{43} = i.$$

The corresponding transformation matrix for spinors is

$$B = exp\left(\frac{i(\sigma_1 p_2 - \sigma_3 p_1)}{2\sqrt{p_1^2+p_2^2}}\,\arctan\frac{\sqrt{p_1^2+p_2^2}}{p_3}\right) = \frac{p + p_3 + \sigma_1 p_2 - i\sigma_2 p_1}{\sqrt{2p(p+p_3)}}.$$

Finally, the transformation of the space-like four-vector $(0, 0, 0, \eta) \to (p_0, p_1, p_2, p_3)$ is performed by the matrix

$$A = 1 + \frac{iS_{3\mu}p^\mu}{\eta} + \frac{(S_{3\mu}p^\mu)^2}{\eta(p_3+\eta)}. \qquad \text{(A.2)}$$

The related transformation matrix for spinors is given by

$$B = \frac{\eta + p_3 - \sigma_3 p_0 - i\sigma_2 p_1 + i\sigma_1 p_2}{\sqrt{\eta(\eta + p_3)}}. \tag{A.3}$$

References

[1] Gentile G Jr 1940 *Nuovo Cimento* **17** 493
[2] Wigner E P 1950 *Phys. Rev.* **77** 711
[3] Green H E 1953 *Phys. Rev.* **90** 270
[4] Greenberg O W and Messiah A M L 1965 *Phys. Rev.* B **138** 1155
[5] Kamefuchi S and Takahashi Y 1962 Nucl. Phys. **36** 117
Ohnuki Y and Kamefuchi S 1982 *Quantum Field Theory and Parastatistics* (Tokyo: University of Tokyo Press)
Greenberg O W 1990 *Phys. Rev. Lett.* **64** 705
Greenberg O W 1991 *Phys. Rev.* D **43** 4211
Okubo S 1994 *J. Math. Phys.* **35** 2785
Palev T and Stoilova S 1994 *J. Phys. A: Math. Gen.* **27** 977
Meljanac S, Stoic M and Mileković M 1998 *Mod. Phys. Lett.* A **13** 995
Jing S and Nelson Ch A 1999 *J. Phys. A: Math. Gen.* **32** 401
Mashkevich S 1998 *Jad. Fiz.* **61** 1864
Meljanac S, Mileković M and Stojić M 1999 *J. Phys A: Math. Gen.* **32** 1115
[6] Gol'fand Yu A and Likhtman E P 1971 *JETP Lett.* **13** 452
Ramond P 1971 *Phys. Rev.* D **3** 2415
Neveu A and Schwartz J 1971 *Nucl. Phys.* B **31** 86
Volkov D V and Akulov V P 1972 *JETP Lett.* **16** 621
Volkov D V and Akulov V P 1973 *Phys. Lett.* B **46** 109
Wess J and Zumino B 1974 *Nucl. Phys.* B **70** 39
Wess J and Zumino B 1974 *Nucl. Phys.* B **78** 1
[7] Fayet P and Ferrara S 1987 *Phys. Rep.* C **32** 250
Fayet P and Ferrara S 1985 *Supersymmetry in Physics* (Amsterdam: North-Holland)
West P 1986 *Introduction to Supersymmetry and Supergravity* (Singapore: World Scientific)
[8] Witten E 1981 *Nucl. Phys.* B **188** 513
Witten E 1982 *Nucl. Phys.* B **202** 253
[9] Rubakov V A and Spiridonov V P 1988 *Mod. Phys. Lett.* A **3** 1337
[10] Beckers J and Debergh N 1990 *Nucl. Phys.* B **340** 767
Beckers J and Debergh N 1991 *J. Math. Phys.* **32** 1808
[11] Durand S and Vinet L 1990 *J. Phys. A: Math. Gen.* **23** 3661
Andrianov A A, Ioffe M V, Spiridonov V P and Vinet L 1991 *Phys. Lett.* B **272** 297
Beckers J, Debergh N and Nikitin A G 1992 *J. Math. Phys.* **33** 152
Beckers J, Debergh N and Nikitin A G 1992 *Mod. Phys. Lett.* A **7** 1609
Beckers J, Debergh N and Nikitin A G 1993 *J. Phys. A: Math. Gen.* **26** L853
[12] Beckers J and Debergh N 1993 *J. Mod. Phys.* A **8** 5041
[13] Jarvis P D 1978 *Aust. J. Phys.* **31** 461
[14] Nikitin A G and Tretynyk V V 1995 *J. Phys. A: Math. Gen.* **28** 1665
[15] Haag R, Lopuszanski J T and Sohnius M F 1975 *Nucl. Phys.* B **88** 61
Wess J and Zumino B 1974 *Phys. Lett.* B **49** 52
Wess J and Zumino B 1974 *Nucl. Phys.* B **70** 39
[16] Wigner E P 1939 *Ann. Math.* **40** 149
[17] Gelfand I M Minlos A P and Shapiro Z Ja 1960 *Representations of the Rotation and Lorentz Group* (Moscow: Fizmatgiz)
[18] Ryan C and Sudarshan E C C 1962 *Nucl. Phys.* **36** 177
[19] Dobrev V K *et al* 1977 *Harmonic Analysis on the n-Dimensional Lorentz Group and Its Application to Conformal Quantum Field Theory (Lecture Notes in Physics 63)* (Berlin: Springer)
[20] Rumer Yu B and Fet A N 1970 *Theory of Unitary Symmetry* (Moscow: Nauka)